Euler and the Multiplication Formula for the Γ-Function.

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We show that an apparently overlooked result of Euler from [*E*421] is essentially equivalent to the general multiplication formula for the Γ-function that was found by Gauss in [*Ga*28].

1 Introduction

The interpolation of the factorial by the Γ-function was found nearly simultaneously by Bernoulli [*Be*29] and Euler in 1729 [*E*19] and is without any doubt one of the most important functions in mathematics. Most of its basic properties were discovered by Euler, who also gave the definition that is nowadays often used to introduce the function [*E*675][1]

$$\Gamma(x) := \int\limits_0^\infty e^{-t}t^{x-1}dt \quad \text{for} \quad \mathrm{Re}(x) > 0.$$

However, a good understanding of Γ as a meromorphic function of its argument could of course only be achieved after Gauss in the 19th century; the now universally adopted notation stems from Legrendre [*Le*09][2].

One of the fundamental properties of the Γ-function is so-called *multiplication formula* that reads, in the modern notation

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{x-\frac{1}{2}}}\cdot\Gamma(x). \tag{1}$$

[1] Opera Omnia, p. 220

[2] See p. 477

For $n = 2$ one obtains the *duplication formula* that is usually ascribed to Legendre [*Le*26].

The multiplication formula was first proven rigourously by Gauss in his influential paper [*Ga*28] on the hypergeometric series, in which he also gives a complete account of the function $\Pi(x) := \Gamma(x+1) = x!$. Gauss cited Euler's results very often, but apparently he was not aware of the lesser-known paper [*E*421] of Euler. In that paper Euler presents a formula that is essentially equivalent (1), as we will explain now.

1.1 The function $\left(\frac{p}{q}\right)$

In §3 of [*E*321] and §44 of [*E*421][3], two papers that were both published in 1766, Euler studies properties of the function

$$\left(\frac{p}{q}\right) := \int_0^1 \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}.$$

In his notation the variable n is left implicit, and Euler shows the nice symmetry property

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

Of course, by the substitution $x^n = y$ this function is just the Beta-function in disguise:

$$\left(\frac{p}{q}\right) = \frac{1}{n}\int_0^1 y^{\frac{p}{n}-1}dy(1-y)^{\frac{q}{n}-1} = \frac{1}{n}B\left(\frac{p}{n},\frac{q}{n}\right), \tag{2}$$

where the Beta function is defined as

$$B(x,y) = \int_0^1 t^{x-1}dt(1-t)^{y-1} \quad \text{for} \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

Euler implicitly assumes p and q to be natural numbers in $\left(\frac{p}{q}\right)$, but this restriction is of course not necessary.

[3] Opera Omnia, p. 343

One of the early discoveries of Euler [E19] was that the Beta-integral reduces to a product of Γ-factors:

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}.$$

This result is also given in the supplement of [E421][4].

1.2 The reflection formula

Euler's version of the reflection formula for the Γ-function,

$$\frac{\pi}{\sin \pi x} = \Gamma(x)\Gamma(1-x),$$

can be found in §43 of [E421] and reads

$$[\lambda]\cdot[-\lambda] = \frac{\lambda\pi}{\sin \pi\lambda}$$

where $[\lambda]$ stands for $\lambda!$, that is $\Gamma(1+\lambda)$.
If one applies the reflection formula for $x = \frac{i}{n}$, $i = 1, 2, \cdots, n-1$ we obtain

$$\begin{aligned}
\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right) &= \frac{\pi}{\sin\frac{\pi}{n}} \\
\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right) &= \frac{\pi}{\sin\frac{2\pi}{n}} \\
\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{n-3}{n}\right) &= \frac{\pi}{\sin\frac{3\pi}{n}} \\
\dots &= \dots \\
\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right) &= \frac{\pi}{\sin\frac{(n-1)\pi}{n}}
\end{aligned}$$

Multiplying these equations together gives our first auxiliary formula

$$\prod_{i=1}^{n-1}\Gamma\left(\frac{i}{n}\right)^2 = \frac{\pi^{n-1}}{\prod_{i=1}^{n-1}\sin\left(\frac{i\pi}{n}\right)}$$

Our second auxiliary formula is

$$\prod_{i=1}^{n-1}\sin\left(\frac{i\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

[4] Opera Omnia, p. 354

which is a nice exercise and which was certainly well-known to Euler. For, in [$E562$][5] he states the more general formula

$$\sin n\varphi = 2^{n-1}\sin\varphi \sin\left(\frac{\pi}{n}-\varphi\right)\sin\left(\frac{\pi}{n}+\varphi\right)$$
$$\sin\left(\frac{2\pi}{n}-\varphi\right)\sin\left(\frac{2\pi}{n}+\varphi\right)\cdot \text{etc.}$$

The product has n factors in total. Therefore, dividing by $\sin\varphi$, taking the limit $\varphi \to 0$, and then using $\sin\left(\frac{\pi(n-i)}{n}\right) = \sin\left(\frac{i\pi}{n}\right)$, we, having divided by 2^{n-1}, arrive at the second auxiliary formula.

The first and the second auxiliary formula were also given by Gauss in [$Ga28$] and are crucial in his proof of the multiplication formula. Combining them and taking the square root, we obtain the beautiful formula

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}. \tag{3}$$

This formula was also found by Euler in [$E816$][6]. Euler states it as

$$\int\limits_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{1}{n}}\int\limits_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{2}{n}}\cdots\int\limits_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{n-1}{n}} = \frac{1\cdot 2\cdot 3\cdot(n-1)}{n^{n-1}}\sqrt{\frac{2^{n-1}\pi^{n-1}}{n}}.$$

2 Euler's version of the Multiplication Formula

In §53 of [$E421$][7] Euler gives the formula

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

As before, $[\lambda]$ is Euler's notation for the factorial of λ, so that $\left[\frac{m}{n}\right] = \Gamma\left(\frac{m}{n}+1\right)$. Euler assumes m and n to be natural numbers, but is easily seen that we can interpolate $1\cdot 2\cdot 3\cdots(m-1)$ by $\Gamma(m)$. Therefore, let us assume x to be real

[5] Opera Omnia, p. 514
[6] Opera Omnia, p. 483
[7] Opera Omnia, p. 348

and $x > 0$ and let us write x instead of m in the above formula. Further, using (2), Euler's formula reads

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{n-x}\Gamma(x)\frac{1}{n^{n-1}}B\left(\frac{1}{n},\frac{x}{n}\right)B\left(\frac{2}{n},\frac{x}{n}\right)\cdots B\left(\frac{n-1}{n},\frac{x}{n}\right)}.$$

Expressing the B-function in terms of the Γ-function and after some simplification under the $\sqrt[n]{}$-sign we find

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{1-x}\Gamma(x)\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+1}{n}\right)}\cdot\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+2}{n}\right)}\cdots\frac{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+n-1}{n}\right)}}.$$

Now, let us simplify this by bringing all Γ-functions of fractional argument to the left-hand side. We will find

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right).$$

The product on the right-hand side, $\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$ was evaluated in (3) and we obtain

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\sqrt{\pi^{n-1}\frac{2^{n-1}}{n}} = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x}\Gamma(x).$$

Thus, we arrived at the multiplication formula (1).

3 Summary and Conclusion

From the above sketch it is apparent that in [*E*421] Euler had a result that is essentially equivalent to the multiplication formula for the Γ-function. He expressed it in terms of the symbol $\left(\frac{p}{q}\right)$, which is in modern notation is the Beta-function. One may wonder why Euler did not express his result in terms of the Γ-function itself. Reading his paper it becomes clear that his main motivation was to express the factorial of rational numbers in terms of integrals of *algebraic* functions, and the formula of Euler fulfills this purpose. For the same reason he probably did not replace $1 \cdot 2 \cdot 3 \cdots (m-1)$ by $\Gamma(m)$. It appears that Euler was aware that the proofs he indicated in [*E*421] were not completely convincing. He expressed that with characteristic honesty in a concluding SCHOLIUM:

Hence infinitely many relations among the integral formulas of the form

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = \left(\frac{p}{q}\right)$$

follow, which are even more remarkable, because we were led to them by a completely singular method. And if anyone does not believe them to be true, he or she should consult my observations on these integral formulas[8] *and will then hence easily be convinced of their truth for any case. But even if this consideration provides some confirmation, the relations found here are nevertheless of even greater importance, because a certain structure is noticed in them and they are easily generalized to all classes, whatever number was assumed for the exponent n, whereas in the first treatment the calculation for the higher classes becomes continuously more cumbersome and intricate.*

The history of the Γ-function is long and complex and apparently not at all complete. We hope that this note provides some motivation to go carefully through other papers by Euler and other mathematicians of the past, not only because they make a good reading, but also to find further results, maybe stated in unfamiliar notation[9], that were proven rigorously by their successors. This will certainly be of interest for anyone studying the history of mathematics.

References

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[8] Here Euler refers to his paper [*E*321].

[9] This might be the reason, why even Krazer and Faber, who wrote the foreword to the Opera Omnia containing Euler's work on integrals (Series, Volume 19, p. LXII) and also discussed [*E*421], missed that Euler's result is essentially equivalent to the multiplication formula.

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